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Perspectives in Mathematics

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1. Introduction

When I was asked to contribute to a symposium on future directions in science, held at the Universitat Autònoma de Barcelona last November, my excitement at the invitation was tinged with a certain diffidence. For I am never comfortable making predictions; I am a compulsive non-gambler, and treat the forecasts of others, as I would treat my own, with profound scepticism.

In the event, I was, to my deep regret, unable to attend that symposium and therefore to deliver my invited address; an accident of geography placed me some 20,000 kilometers from Barcelona at the time of the symposium and this accident was deemed to constitute, despite the marvels of modern technology, a powerful obstacle to my attendance. However, my disappointment was tempered by the invitation of my very good friend Professor Manuel Castellet to come to Barcelona at this time to join in the celebrations of the 10^{th} anniversary of the founding of the Centre de Recerca Matemàtica; and I decided to take advantage of the opportunity thus provided to take up again the challenge of trying to discern the shape of mathematics and the nature of mathematical activity in the years to come. Is this not, after all, an appropriate theme for the celebration of a decade of successful research under the auspices of this fine Centre and its inspiring Director?

For I am fully aware that there *is* a challenge in the title of my address which I actually relish, despite my reluctance to indulge in rash prediction. I can look at the situation in mathematics today, and in mathematics education, and seek to identify the actual dominant trends. I can then leave to others the formulation of detailed forecasts for the future. This, then, is broadly what I have done; but I am aware that, where I attempt to characterize the trends in mathematics itself, I find myself striking an optimistic note which I am far from really feeling. For I am frankly very pessimistic with regard to the short-term future of education in general (I understand 'short-term' to mean a period not exceeding 20 years), and therefore must be apprehensive for the development, in the somewhat longer term, of mathematics, since the health of mathematics must depend on the support of a vibrant educational system.

Naturally, in speaking of trends, in mathematics or mathematics education, I confine myself largely to the societies I know well, that is, to the United States, Canada and Western Europe. However, the ending of the Cold War has had a certain homogenizing effect on human societies. One does not have to agree with Fukuyama's thesis that history has reached its end - I certainly do not - to acknowledge that, for the time being, capitalism and the free market economy will be largely favored over systems of public ownership of the means of production and exchange. It is clear, then, that certain distinctive features of education and intellectual life which we associate with the sometime socialist states of Eastern Europe will, temporarily at any rate, disappear, so that my comments may turn out to be applicable over a broader range than I explicitly claim for them.

Thus I plan in the next section to discuss trends in university mathematics education, with the United States as my prime example. In Section 3 I will look at developments in mathematics itself; and, in the final section, I will allow myself a few speculations as to how these trends could affect the future of both education and mathematics. These speculations are intended to do no more than provide a basis for the discussion of these issues by those less reluctant than I to gaze into the crystal ball and announce what patterns they discern.

2. Trends in university education

I have been a university teacher for over 45 years; yet never before have I known such intense concern among my colleagues about the quality and effectiveness of their teaching. Indeed, it would be fair to say that discussion of these issues has now displaced the discussion of new mathematics as the most frequently heard topic among mathematicians. (This is especially significant since, in times past, such a topic was not even in second place, being heard far less frequently than what may be termed 'mathematical gossip'.)

Our concern centers on two principle matters: (a) the teaching of undergraduate majors, and (b) the preparation of future mathematicians. Of course, these are related themes, and it is probably true that, for many academic mathematicians, their anxiety over (a) is due to their even greater, and more immediate, anxiety over (b). Indeed, it is noteworthy that the American Mathematical Society, which, since the (misconceived!) founding of the Mathematical Association of America nearly 80 years ago, has determinedly avoided any responsibility for the teaching of mathematics, perceiving its role exclusively to be that of supporting mathematical research, has now joined with the MAA and other concerned representative bodies to examine, and seek to improve, undergraduate curricula and the delivery of mathematical instruction.

The undergraduate major in American universities is in sharp decline - the situation would be even worse if joint computer science/mathematics majors were not available at many of our universities. Our brightest students have long since abandoned physics as their first choice; they are now abandoning mathematics, too.[1] They are showing a preference for law, medicine or business studies, perceiving the job prospects to be far brighter in these fields. What is significant here is not any change which may have occurred in relative job prospects - it is that the students are applying such very pragmatic criteria to their choice of major, and are apparently quite unimpressed by any argument relating to the excitement and intellectual stimulation likely to accompany one course of study rather than another.

The shortage of mathematics majors naturally affects the intake into our graduate schools of mathematics, so that today not even the most prestigious graduate schools can ignore the cold wind of change blowing from our centers of undergraduate education - hence, as I have suggested, the new interest of

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[1]Many of the statements in this address are of a statistical nature. Fortunately, such statements, unlike mathematical propositions, are not invalidated by the existence of the occasional counterexample.

the faculty of our elite institutions in problems of undergraduate teaching. It is cold comfort that, in other scientific and engineering disciplines, the situation in our graduate schools is even worse; it is common knowledge that many of those schools survive on their intake of foreign, largely oriental, students.

However, it would be grossly misleading to give the impression that the real problems of undergraduate education are confined to, or even concentrated in, the mathematics major. Though not so engrossing or ubiquitous a topic of conversation among the mathematical faculty, the problems associated with trying to teach mathematics as a service course are surfacing ever more insistently. Let us discuss some of these.[2]

Students seem to be generally less well prepared technically than they used to be. Thus it is usually unwise to assume that students can accurately, intelligently and efficiently use mathematical techniques and ideas to which they have been exposed in their previous mathematics courses. Students frequently complain of the 'irrelevance' of the mathematics they are being asked to learn; by this they may mean that they see no application to the real world, or, more frequently, they may simply mean that they cannot conceive that, in their chosen or intended careers, they could possibly require this particular piece of mathematical knowledge.

I contend that these complaints are usually specious. Students feel inclined to complain if they are not understanding and are performing unsatisfactorily. It is then human nature to try to shift responsibility and the charge of irrelevance is an attempt to do so. Students who are doing well very seldom

[2]The reader will notice that most of these problems are also relevant to the undergraduate major.

seem to be affected by the virus of irrelevance. However, the nature of the complaint is highly significant. For it indicates that the student sees no benefit in learning, and understanding, some abstract ideas (such as the conceptual basis of the calculus or the nature of a group) unless the acquisition of such ideas increases his, or her, marketability. This attitude on the part of the student implies an unawareness of the fundamental purpose of education and the false belief that it is better to undergo training than education.

Perhaps we have identified here one of the most important factors affecting the future of mathematics. If students only want to be trained in marketable, 'relevant' skills, they will certainly not prepare themselves for careers as mathematicians, or even for careers in which they will be called upon to use mathematics in intelligent ways. For such a preparation requires that one develop a thirst for knowledge and understanding and an intelligent curiosity; it also requires that one tackles intellectual challenges with a zest which rivals that shown for any favored pursuit, inside or outside the classroom.

We should add that a closely related phenomenon of today's university campuses is the lack of idealism among the students. This is in sharp contrast to the spirit which pervaded American institutions of higher education 30 years ago. I recall the 'bad old days' of student protest in the late sixties. Although I often disagreed with the students then, I would far prefer to be dealing with a group of students motivated by an idealism, however flawed in practice, than a group of present-day apparatchiks who make it plain that they are only there to get a qualification, a certificate, not to become educated persons.

The problems which we are here discussing naturally far transcend the difficulties of effective mathematics education, even though they conspicuously manifest themselves in those difficulties. They are societal problems and they reflect the value-system currently adopted by the most financially successful members of that society, that is, by the most influential groups and individuals in our 'advanced' democracies. Thus it will be through future (let us hope, imminent) changes in the predominant values of our society that encouragement will be given to students once again to value education, and to universities to play their proper role in nourishing the educational roots of society. Plato and Spiro Agnew had one idea in common (and probably only one): both recognized that a good, strong educational system was a threat to the Establishment. Plato saw this as a good thing; Agnew saw it as a danger to be combated. There are grounds for hoping that the Platonic view is again beginning to gain the upper hand - but there are no grounds whatsoever for complacency. No one here would, I am sure, regard, the spirit which imbues the CRM as characteristic of modern attitudes.

3. Trends in mathematics

Consistently with my usual cautious approach, I am happier discussing actual trends in mathematics than speculating on future trends. However, I take comfort from the fact that even Michael Atiyah, in his very enlightening address to the 1976 Karlsruhe Conference on Mathematics Education [A], devoted most of his remarks to a powerful analysis of what were the characteristic features of contemporary mathematics. It will be recalled that he drew especial attention to the fact that we now, typically, study functions of several variables rather than functions of a single variable, as in 19th century mathematics; certainly this feature of our mathematics remains as prominent as ever. One may also notice - without surprise - the enormous impact the computer has had and continues to have on mathematics itself. It has brought certain areas of mathematics into prominence - indeed, in a few striking cases, such as the study of computational complexity, into very existence - and it has changed the nature of others. It should have had a striking impact on our teaching of mathematics, in at least two respects, but this impact is not yet as noticeable as one might have wished.

In the first place, we should have seen, at least in the advanced industrial nations, more use of the computer as a tool in the teaching of mathematics. I have especially in mind the use of computer graphics to illumine fundamental concepts of the calculus and to bring to life in very vivid and memorable form such areas of mathematics as the solvability of systems of linear equations and Euler's method for solving the differential equation $\frac{dy}{dx} = f(x, y)$ with given initial condition, where f satisfies a Lipschitz condition. In the second place, we should also have seen already a decisive reassessment of the importance of certain items of the traditional curriculum. Thus computational techniques (all the way from the traditional hand-algorithms of elementary arithmetic to techniques of integration) should have been very substantially de-emphasized; and questions of *whether* sequences or series converge should have yielded in importance to questions of *how rapidly* they converge, that is, to questions of whether they serve to provide useful approximations to their limits.

On the other hand, the prognostications of some computer scientists on the likely, and the desirable, effect of the availability and increased sophistication of computers on the curriculum are not being verified, and for good reason. Thus James Frauenthal in SIAM News in 1980 predicted that the computer would permanently displace mathematical analysis from the centre of the stage, in both teaching and research (along with number theory and topology) and would create a pattern of mathematical activity which, by the year 2025, would be utterly different from that of today. I believed then, and I still believe, that Frauenthal was wrong,[3] because he did not appreciate the close relationship between effective computer use and familiarity with the principles of mathematical analysis, and because he was unable to envisage the vast growth in the diversity of mathematical application.

In fact, we are, I hope and believe, in the process of abandoning two of the most notorious, and most dangerous, false antitheses of mathematics in practice in this century - and especially prominent in the period 1945-1985. I take great pride in the fact that I highlighted these erroneous views in my own address to that same Karlsruhe conference [H1]. I refer to the antitheses

{ teaching vs. research; and
{ pure vs. applied mathematics

As to the former, I have already indicated why many of today's outstanding researchers are showing great concern for the teaching of mathematics; and I would refer the reader to my article [H3] for a sustained argument, from a different standpoint,

[3]See also the critique by Gail Young of Frauenthal's thesis, in his Introduction to [HY].

for the complementarity, rather than the conflicting nature, of these two responsibilities of the mathematician. Surely it is not unreasonable to hope that this spectre, given such undue prominence by the hostile propaganda of Morris Kline [K], has now been laid to rest!

It is the second false dichotomy above to which I want now to give attention, since it is my case that the disappearance of this artificial distinction is one of the principal distinguishing marks, and achievements, of contemporary mathematics, and will continue to enrich our science in the foreseeable future. I was aware of a changing attitude already in the late 1970's (see [H2]); but a list of the areas of mathematics treated by the speakers at the Conference on New Directions in Applied Mathematics [HY] surely reinforces my position. Thus we find featured

Ordinary and partial differential equations (of course!)
Combinatorics
Commutative algebra
Theory of jets (differential geometry)
Algebraic geometry
Lie groups and Lie algebras
Differential topology
Algebraic topology, fibre fundle theory
Deformation of complex structures
Singularity theory, chaos theory
Functional analysis

Indeed, it is fair to add that had a specialist in coding theory been able to accept our invitation to participate, one could have added finite group theory, number theory and the theory of finite fields to the list[4]! The reader will note that there are many topics mentioned above which would, even 20 years ago, have been regarded as ineffably pure. One sees here just why, and how fundamentally, Frauenthal's prediction is wrong. An emphasis on applied mathematics does not limit the areas of mathematics suitable for study; it broadens the areas of mathematics suitable for application. That is part of the wonder of mathematics - it is its nature to be applicable, provided, of course, that it is intrinsically worthwhile mathematica.[5] As Gail Young has written, in his own attempt at predicting the future in his Introduction to [HY], 'By the end of the century, the temporary overemphasis on pure mathematics was completely gone and the traditional interconnections between pure mathematics and applications restored'. I would only add that, plainly, the traditional interconnections are being strengthened and new connections established.

Surveying the current research scene, one notices that the convergence of pure and applied mathematics, the re-establishing of strong links between mathematics and physics - but by no means confined to those areas of mathematics traditionally associated with physics - is part of a strong trend toward reaffirming the unity of mathematics itself. Thus I am emboldened to predict that a continuing feature of mathematical research in the coming decades will be in the direction of establishing links between its various parts. Broadly speaking, one may say that the spectacular progress in mathematics in

[4]And if DNA had been on our agenda we would surely have required an expert in the theory of knots.

[5]Today, category theory and logic are being recognized as essential tools in theoretical computer science.

the years 1950-80 was in the development of autonomous disciplines. Most practitioners were highly skilled specialists and worked in fairly narrowly defined areas of mathematics - finite group theory, homotopy theory, commutative ring theory, non-commutative ring theory, ordinary differential equations, partial differential equations, etc. One may characterize such research as *vertical research*, building very tall, very refined, but rather narrowly based structures. By contrast some of the most exciting research today is *horizontal research*, establishing important and hitherto unsuspected connections between these structures. Examples are invariant gauge theory, cohomology theory and elliptic differential equations, homological algebra and the decidability of machines, group theory and the study of homeomorphisms of subsets of Euclidean space. No criticism is here implied or intended of the early period of vertical progress - one can only establish links, and one should only seek to establish links, when the structures to be linked have well-understood analytical properties. But it is most gratifying that the era of horizontal progress is upon us, a tribute to the significance of the specialized work done earlier. Moreover, the implication for our teaching of mathematics is obvious - and important.

4. A glimpse into the future

Let me, at first, adopt an optimistic viewpoint as to what the future will bring. I have already described my picture of the future of mathematics itself, insofar as I can perceive it. What kind of education should one find, to provide the mathematicians of the future and to fulfill the other, manifold functions of education in a modern, enlightened society?

Surely we will not be laying any emphasis on the teaching of those mechanical skills which (naturally!) machines perform far better and faster. These range, as earlier indicated, from elementary arithmetic to techniques of integration and the solution of systems of linear equations and linear differential equations with constant coefficients. Above all, we will be teaching for understanding and appreciation rather than the acquisition of speed and accuracy in the execution of algorithms; and our methods of testing our students will reflect this new emphasis. This last point is crucial - so long as we inflict on the students test questions which require of them manipulative skill and speed, together with a certain flair for pattern recognition, a good memory, and nothing else, it is pointless for us to protest that, in our teaching, we emphasize the importance of proof, of problem-solving strategies, and other characteristic aspects of effective mathematical activity. Let us be clear; in the United States the tests drive the curriculum, in the sense of the actual teaching curriculum, at every level. It should be the other way round, but it is not; and it would be fatuous to adopt so optimistic a stance as to imagine that this might change, too.

Society must provide a support system for effective education. Hence implicit in this rosy spectacle for the future of mathematics education is the assumption that our citizens have acquired an appreciable mathematical literacy. They must be aware of the role of mathematics not only in modern industry, but also in intelligent living. This implies, for instance, an understanding of orders of magnitude, of approximate arithmetic and estimation, of the principles of probability and statistics, and of the nature and implications of continuous change. Also, since rational decision-making is going to become ever more important if real, effective democracy is to be achieved, we must hope that our educational system will have armed our people with the ability to assess the validity of the arguments presented to them and to detect the superficial, the specious and the downright false. Wouldn't it be wonderful if the slogan 'Fairness in Advertising', instead of representing the consumers' defense against malpractice, expressed the enlightened self-interest of the producer?

However, honesty compels me to admit that I am not very sanguine about the future; the picture I have been painting above represents, at best, the aspirations which mathematicians and educators will continue to carry with them far into the 21st century. As I survey the scene today, especially the prevailing value-system in the United States and other advanced industrial societies, and the forces unleashed by the elimination of (so-called, but spurious) communist regimes in Eastern Europe, I find myself wondering if we are not entering a new cultural Dark Ages. May it be that the spirit of education and scholarship will be kept alive only in a few centres of enlightenment, akin to the monasteries of the old Dark Ages, while, outside their walls, people seek only the personal advantage of themselves and their immediate families? Materialism seems to be the dominant religion of our time, as Commander Jacques Cousteau deplored at the World Summit on the Environment in Rio in 1992; and this is having not only an unfortunate ecological effect but is also contributing to a decline in intellectual standards. To advance the cause of education we need to effect a radical change in the prevailing value system and to overcome many current prejudices against science (manifested, for example, in the campaign conducted by Bryan Appleyard and others against the hubris of Stephen Hawking in presuming to 'read the mind of God').

If, however, we can make the necessary changes in popular attitudes, then mathematics is sure to prosper, and to continue to bring long-term benefit to society and enlightenment, understanding and spiritual enrichment to those who make it, those who teach it, and all those who make effective contact with it. It is by this criterion that the achievements of CRM should be judged, and, quite clearly, should be judged to be successful.

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